

# Design and Implementation of a Division Tool for Elementary Math Education

by

Elizabeth M. Simon

Submitted to the Department of Electrical Engineering and Computer  
Science

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## Abstract

This thesis presents the design, implementation, and testing of a novel division tool for elementary math education. Division is a concept with which many elementary students struggle. The division tool, which is an addition to the pen-based wireless classroom interaction system called Classroom Learning Partner, provides an interactive visual model for the division process. This model leverages students' understanding of the array as a model for multiplication in order to help students develop a strong understanding of division. The tool was tested with students in the Boston area. Observation and feedback, from both students and teachers, indicate that the tool has great potential for improving students' understanding of division.

Thesis Supervisor: Dr. Kimberle Koile  
Title: Research Scientist, MIT CECI



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# Chapter 1

## Introduction

### 1.1 Motivation

Many elementary school students struggle with the concept of division. When first approaching the subject, division can be much more complicated than multiplication. There are many numbers and concepts to consider at once: the dividend, the size of each group, the number of groups, and a changing total computed while performing the calculation. Students have a difficult time thinking about the total and the groups they are forming simultaneously, which makes division particularly confusing [1]. With the problem 58 divided by 7, for example, the student will need to figure out how many groups of 7 can be made from 58. A common strategy is to subtract groups of 7 until the total left changes from 58 to 2, which is the remainder. This type of strategy is shown in Figure 1-1. If the student subtracted 3 groups of 7 so far, in addition to keeping track of the dividend, 58, and the divisor, 7, she must also keep track of the total so far, 21, and the amount left to reach the dividend, 37. It is difficult for a student to think about all of these numbers at once and the relationships between them.

Students are taught to solve division problems using a standard algorithm of repeated subtraction. However, it is often more intuitive to consider counting up groups to the total, especially when first learning about division [1]. Employing a counting up strategy for the above example of  $58 \div 7$ , the student would start with 0 and add

58 ÷ 7 =

8 remainder 2

$$\begin{array}{r}
 7 \overline{) 58} \\
 \underline{-21} \quad 3 \\
 37 \\
 \underline{-21} \quad 3 \\
 16 \\
 \underline{-14} \quad 2 \\
 2 \\
 \hline
 8
 \end{array}$$

Figure 1-1: The repeated subtraction method applied to solve the problem  $58 \div 7$ ; this method of organizing the subtraction is often referred to as the "Big 7"

groups of 7 until reaching 56, then note the remainder of 2. An example of a counting up strategy is illustrated in Figure 1-2. The idea of repeated subtraction is particu-

58 ÷ 7 =

| <u>Total</u>   | <u>Groups of 7 so far</u> |
|----------------|---------------------------|
| 7 + 7 + 7 = 21 | 3 7's                     |
| 21 + 21 = 42   | 3 + 3 = 6 7's             |
| 42 + 7 = 49    | 7 7's                     |
| 49 + 7 = 56    | 8 7's                     |

8 remainder 2

Figure 1-2: An example of the counting up strategy used to solve the problem  $58 \div 7$

larly confusing because of the fact that students struggle to simultaneously consider the part and the whole, i.e. each individual group and the total of all the groups.

As a result, students may have a weak understanding of division, which hinders their performance as they progress to more advanced mathematics. According to a 2012 study, "Analyses of large, nationally representative, longitudinal data sets from the United States and the United Kingdom revealed that elementary school students' knowledge of fractions and of division uniquely predicts those students' knowledge of algebra and overall mathematics achievement in high school, 5 or 6 years later, even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education" [6].

Visual abstractions can be a useful tool for encouraging more efficient computational strategies [1]. Arrays are one example of a useful abstraction found in many elementary math curricula; an array enables students to visualize a product as the area of the array. Arrays, for example, can help students understand how to make use of the distributive property to solve a multiplication problem more efficiently, as seen in Figure 1-3. Our goal is to provide tools that allow students to visualize the concept of division in a way that reflects their intuition and builds on their understanding of arrays. We have successfully developed and used an array tool to teach multiplication in fourth grade classrooms, and we believe that the array model has significant potential for improving students' understanding of division as well.

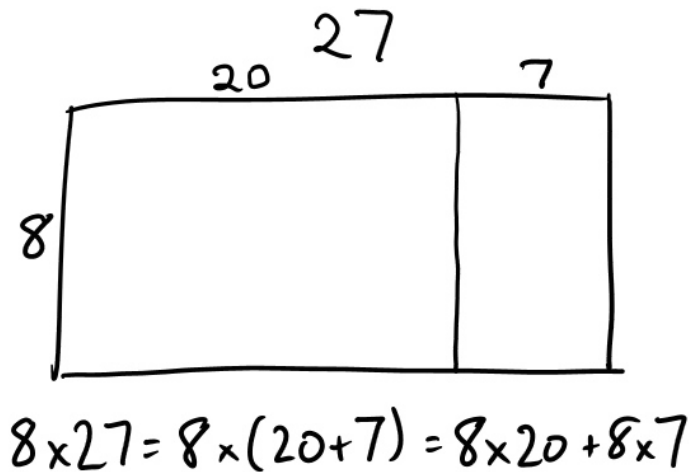


Figure 1-3: An array being used to explain the distributive property for the problem 8 times 27

## 1.2 Overview

This thesis presents a new tool designed to help students develop a strong understanding of division. This tool is part of a software system called Classroom Learning Partner, which our own research group has developed. The system allows students to use a tablet pen to create answers to in-class problems using various software tools that we have developed and wirelessly send their answers to the teacher. Classroom Learning Partner's new division tool described in this thesis builds upon students' understanding of arrays as a model for multiplication in order to help them develop an intuitive understanding of division. The tool helps students visualize the idea of adding groups to reach their total, while also connecting this idea to repeated subtraction, in order to help them understand both ways of looking at the problem. Students can interact with the visual representation while the tool keeps track of the numbers in the problem for them. Figure 1-4 shows an example of the division tool being used to visualize and solve the problem  $58 \div 7$ .

Details of interaction, design rationale, and evaluation are described in the chapters that follow.

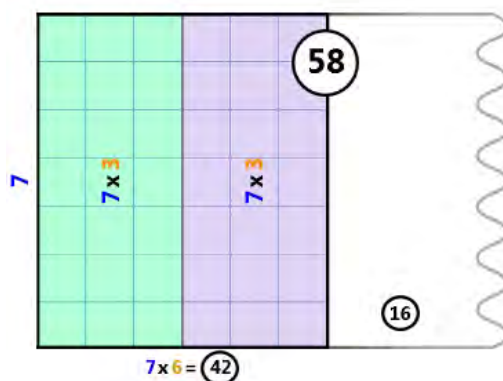


Figure 1-4: The division tool being used to solve the problem 58 divided by 7. In this representation, the 58 in the center is the dividend and the 7 in blue is the divisor. The 42 at the bottom is the total area of the colored portion representing the amount counted up towards the total so far, and the 16 in the right portion of the object is the value remaining to reach the dividend.

## 1.3 Outline

Chapter 2 provides background information on students' understanding of division, the software that this tool was built within, and related work. Chapter 3 provides a more detailed description of the division tool. Chapter 4 describes the process of the design of this tool. Chapter 5 details our experiences with testing the tool. In Chapter 6 we describe automatic analysis of student work using what we call tags. Chapter 7 explores possibilities for future work and summarizes contributions.



# Chapter 2

## Background

### 2.1 Division Strategies

The first stage of this project involved researching how students think about division. For the purposes of this project, we aligned with Fosnot and Dolk's interpretation of students' understanding of division as detailed in their book "Young Mathematicians at Work: Constructing Multiplication and Division" [1]. They describe a variety of strategies students use, starting with very basic strategies that necessitate a lot of counting, and working their way up to more efficient strategies which require a more abstract understanding. Our goal is to provide tools which provide assistance at each level and encourage students to use and understand the more advanced strategies.

Division can be thought of in two ways: building up, which involves starting from zero and adding groups of the appropriate size until the sum equals the desired total, or breaking down, which starts with the total and subtracts groups to reach zero. Breaking down strategies can be more efficient, but also can be more difficult for students to conceptualize because students often have a hard time considering the individual groups and the total simultaneously.

There are two types of division: *quotative* and *partitive*. Quotative problems specify the total and the size of the groups and ask the student to figure out the number of groups. Partitive problems specify the total and the number of groups and ask for the

size of the groups. If someone was dividing 12 cookies into plates of 3, for example, that would be a quotative problem because the size of the groups is known. If someone wanted to divide 12 cookies evenly amongst 4 friends, however, that would be a partitive problem since the number of groups is known. A progression of strategies for each type of problem is outlined below. Any of the strategies below can be applied in either a building up or breaking down manner.

### **2.1.1 Quotative Problems**

#### **1. Counting**

The counting strategy is the most basic type of strategy students use to solve quotative division problems. It involves creating groups with the desired size as specified by the problem and counting the total until it adds up to the desired result. If a student wants to divide 18 people into groups of 3, for example, she could draw groups of 3, counting the total number of items each time, until the total equals 18. The answer will be the number of groups the student has made.

#### **2. Abstract Groups**

This strategy is similar to the previous one except at this stage students do not need to count each item, but instead they consider abstract groups with the size specified in the problem. Using this strategy without tools, a student might draw the number in a circle once for each group and then use addition to determine how many groups are needed to reach the total. To employ this strategy to the problem of  $18 \div 3$  above, a student could add up groups of 3, keeping track of the total and the number of groups, until the total equals 18. The answer will be the number of groups of 3 used to reach 18.

#### **3. Making Use of the Distributive Property**

Once students strengthen their abstract understanding of division, they can make their calculations more efficient by taking advantage of the distributive property and



smaller multiplication facts. If a student wants to assign 84 chairs to tables that seat 6 chairs, for example, she could count up to 84 by first adding in  $9 \times 6 = 54$  and then adding another  $5 \times 6 = 30$  to get  $9 + 5 = 14$ .

Another significant development in students' understanding of division is the realization that they can make use of the place value system when using the distributive property. Students can make their strategies more efficient by using multiples of ten that are easier to add or subtract. In the problem above, for example, the student could add  $10 \times 6 = 60$  with  $4 \times 6 = 24$  to achieve the same answer. Teachers often encourage students to work with "landmark" or "friendly" numbers such as 10 or 5 in order to make their calculations easier.

## **2.1.2 Partitive Problems**

### **1. Trial and error**

A student using a trial and error approach will create groups and count the total to check if it matches the goal, then adjust the groups until the total is correct. This works very well with physical representations such as cubes. If a student is asked to divide 20 cubes into 4 groups, she could start by trying 4 cubes in each group. The student would then count the total, realize that it is less than the goal of 20, and try a larger number. Once the student has found a group size that results in the 4 groups having a total of 20 items, her answer will be the size of those groups.

### **2. Dealing Out**

Dealing out is the process of giving equal amounts of an item to each group in a series of rounds until there are none left, or in the case with a remainder, fewer items than the number of groups. Once students understand dealing out produces equal groups, they can use dealing out as a strategy for partitive problems. Students can deal out physical objects or tallies on a page one by one into the appropriate number of groups until the desired total is reached. This strategy employs the building up

method discussed earlier. If a student was asked to solve a problem involving sharing 24 cookies amongst 6 people, she could draw 6 groups and add one to each group in turn, keeping track of the total. Once the total reached 24, the student could count the number in one group to find her answer. Alternatively, students can employ a breaking down strategy by dealing from a pile or total that has the desired value. To be more efficient, students can deal more than one item at a time.

### **3. Quotative Strategies for Partitive Problems**

Quotative strategies for division tend to be more efficient and easier to comprehend than partitive strategies. Once students realize that quotative and partitive division are equivalent, they can learn to solve partitive problems using quotative strategies.

Arrays are a very useful method for demonstrating the equivalence between quotative and partitive problems. Rows can be used to represent the groups, especially if the problem has an array-like structure such as placing items on a bookshelf. Students can then rotate the array or draw another array with the dimensions reversed, to prove that the quotative equivalent of the problem has the same solution.

## **2.2 Classroom Learning Partner**

The tool described in this thesis is an addition to an existing classroom interaction system known as Classroom Learning Partner, or CLP. CLP employs a model of interaction in which students use a tablet pen to create digital ink inscriptions in an electronic notebook, and the software allows students to share the inscriptions wirelessly with the teacher. Behind the scenes CLP attempts, when possible, to interpret the students' work in order to provide the teacher with feedback about student understanding.

In addition to supporting drawing with a tablet pen, CLP has a variety of tools to

help students create representations of mathematical concepts. The two main CLP tools currently used in mathematics are what we call *stamps* and *arrays* [2][3][4]. The stamp and array tools, as well as the new division tool, are accessible via the buttons on the top command bar, as shown in Figure 2-1. (See [3] and [4] for a detailed description of stamps.)

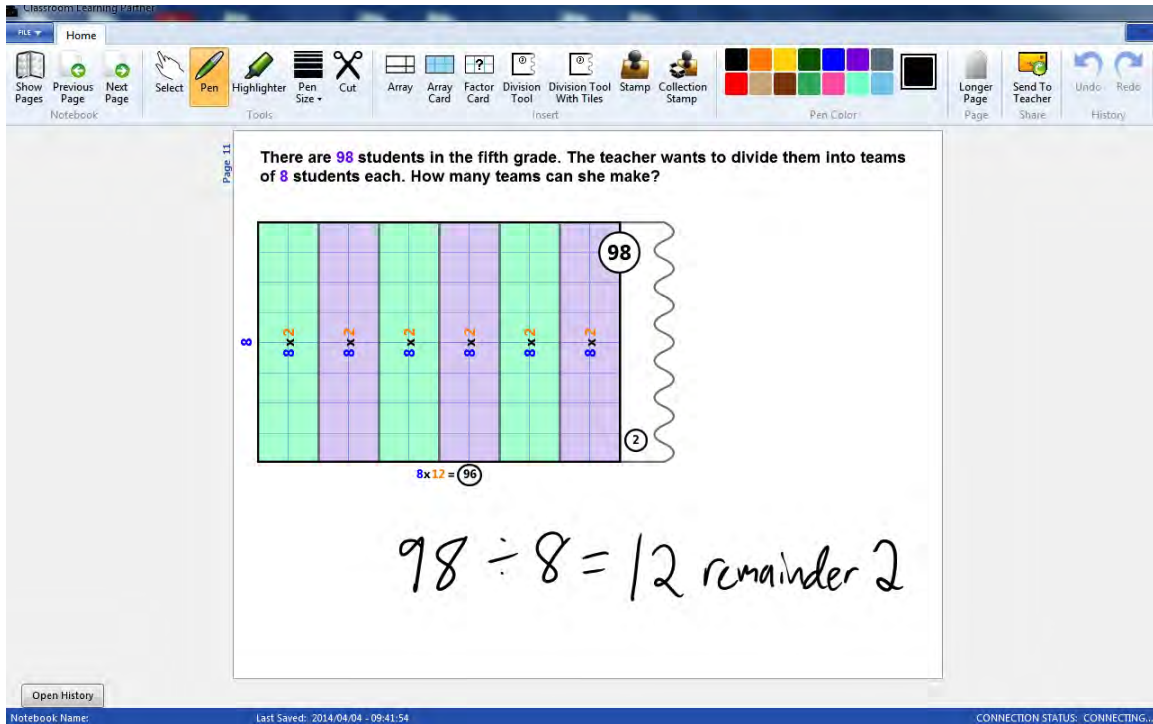


Figure 2-1: CLP being used to solve the problem  $98 \div 8$

## 2.2.1 Stamps

The stamp tool allows a student to draw a representation in a designated location—the body of the stamp—and then create copies of that image. Its use mimics that of a physical stamp. Stamps are especially helpful when students create representations that involve multiple identical images, e.g., 4 groups of 3 objects. In the example shown in Figure 2-2, the student solved the quotative division problem by repeatedly drawing a bag and "stamping" 3 cookies into it until reaching a total of 12. The num-

ber below the stamp indicates the number of items on the stamp; a student enters this value via a number pad.

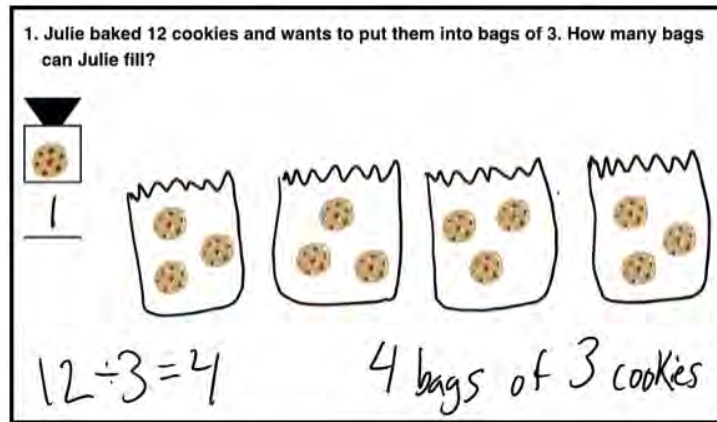


Figure 2-2: An example of stamps being used to solve a quotative division problem

CLP also has a tool called *collection stamps*, which keep track of the total number of other stamp copies placed on top of them. In the example shown in Figure 2-3, the student has created two stamps: a regular stamp to represent a cookie, and a collection stamp to represent a bag. The student solved the partitive division problem by "stamping" 4 bags and then dealing out cookies using the cookie stamp, until there were 12 cookies accounted for.

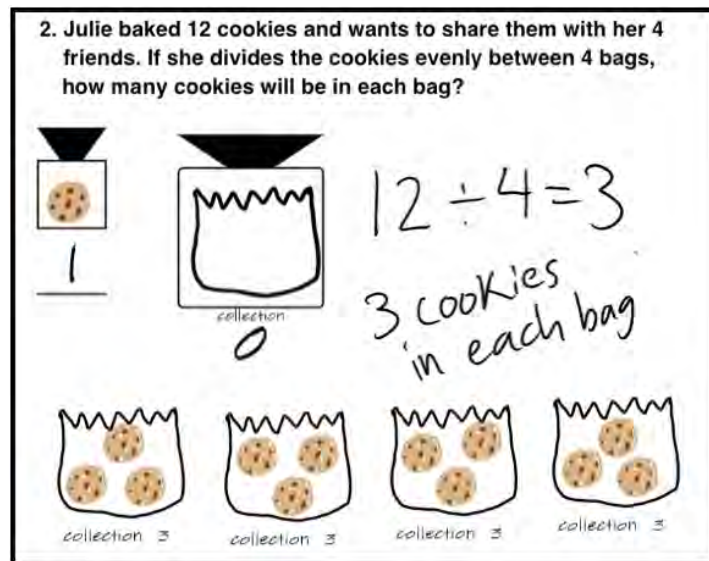


Figure 2-3: An example of collection stamps being used to solve a partitive division problem

Using stamps to solve division problems works well when students are first learning division since the stamps help students visualize the groups. Stamps are not efficient when students move on to larger numbers since it is tedious to manipulate many stamp copies, hence the need for a new division tool.

## 2.2.2 Arrays

Previous work in this project included the implementation of an array tool for CLP. The infrastructure for the array tool, as well as many of the mathematical ideas it utilizes, were a large part of the foundation for this thesis. The goal of the array tool is to help improve students' understanding of multiplication. Students specify the dimensions of an array via a pop-up window accessed with the "Array" button on CLP's command ribbon, and an array appears on the page. Small arrays have a light blue grid. The grid can be toggled on and off via an "adorner", which is an operator we use to perform various actions on objects, accessed by tapping on the object. The grid helps students visualize multiplication as finding the area of an array. Students can create sections in the array by drawing vertical or horizontal lines across it. This action places what we call a divider. Students then label the lengths of the divided segment by tapping on the "?" that appears and using a number pad to enter a label. Arrays in the process of being divided and labeled can be seen in Figures 2-4 and 2-5. The dividers split an array into sections, each of which represents a partial product, as shown in Figure 2-4.



(a) Students draw lines through the array (b) A divider appears near the student's line

Figure 2-4: Students create dividers to break the array up into sections

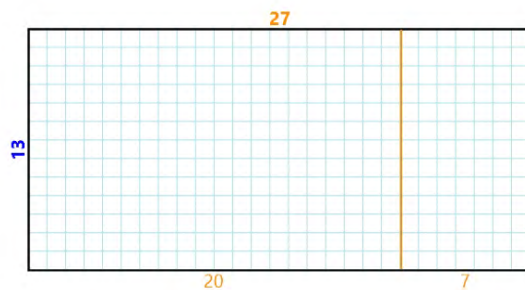
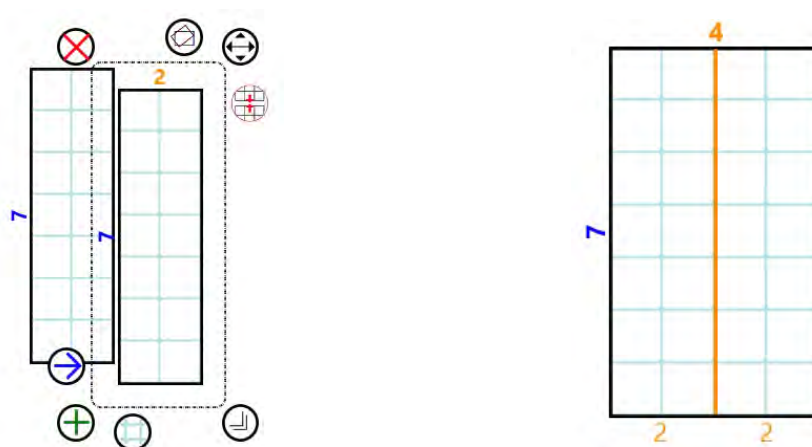


Figure 2-5: Students label the parts of the side of the array by tapping on the "?" and using a pop-up menu

In addition to adding dividers, students can rotate and cut arrays, and snap multiple arrays together. In order to snap two arrays together, students can drag an array using the "snap adorer", an icon which pops up when the array is selected, and align its edge with the edge of another array. If the dimensions match, the arrays will combine into one array with a divider where they were joined, as in Figure 2-6.



(a) Students drag using the snap adorer

(b) The arrays snap together

Figure 2-6: Students can snap arrays together by dragging with the "drag and snap" adorer on the right

Variations on arrays also have been implemented. These variations resemble physical cards that students use in the TERC Investigations math curriculum. The first of these variations is the *array card* which has no labels, as shown in Figure 2-7.



Figure 2-7: An example of an array card

Array cards are useful for placing arrays side-by-side to create larger arrays or overlaying small unlabeled array cards on top of larger labeled arrays to represent breaking large arrays apart, as demonstrated in Figure 2-8. The students can use the tablet pen to write the dimensions of the array cards that make up the larger array.

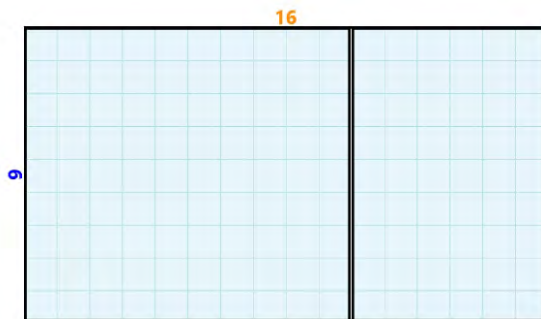


Figure 2-8: Two array cards overlaid on a 9 by 16 array

The second array variation is the *factor card* which asks students to specify the product and one of the factors and leaves one factor unspecified. An example is shown in Figure 2-9. The factor card is a preliminary model for division, as it provides the total and a single factor with one factor unknown, as in a division problem.

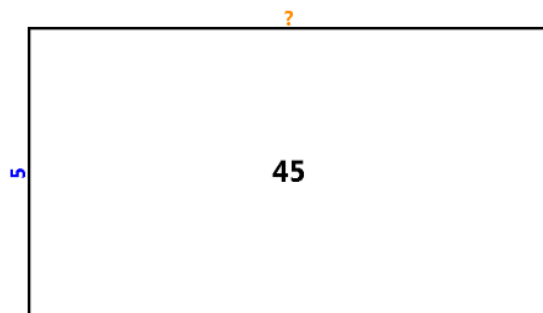


Figure 2-9: A factor card representing the problem of 45 divided by 5

The array tool can be used by students to model division, particularly quotative division. For the problem of  $55 \div 7$ , for example, students can create 1 by 7 arrays and line them up or snap them together to represent the groups of 7. To make use of the distributive property, students can use one array to represent multiple groups, such as a 5 by 7 array to represent 5 groups of 7. The student can then add more groups until the total of 55, or close to 55, is reached. These methods are limited, however, as they do not help a student visualize or keep track of the total amount remaining. We incorporated and improved on this idea in designing and implementing a tool specifically meant for division. The tool, described in Section 3, builds on this array snapping idea, as well as on the factor card and array overlay ideas.

## 2.3 Related Work

There are a variety of educational software systems that offer tools designed to help students develop an understanding of concepts such as division. One example is the educational software system called Dreambox, which runs on the iPad ([www.dreambox.com](http://www.dreambox.com)). Dreambox has a selection of math tools that use arrays to model multiplication and division. One tool, shown in Figure 2-10, represents division problems as arrays with one side marked with a question mark. Students can type smaller equations that help solve the larger problem, and an animated spaceship below the array shoots a laser to represent a division in the array. This model



helps students visualize the concept of division and helps them understand the idea of breaking up the problem into easier chunks, but doesn't allow them to physically interact with the visual model.

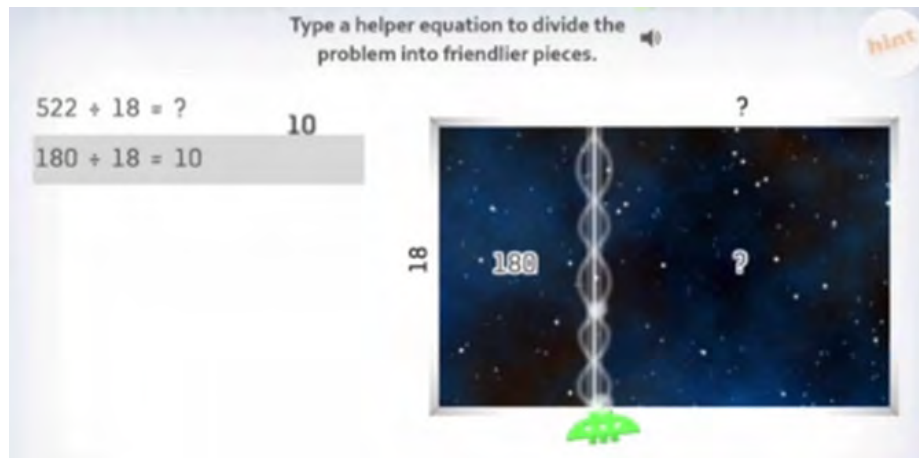


Figure 2-10: The dreambox division tool

The National Library of Virtual Manipulatives (<http://nlvm.usu.edu/>) also has a tool to help students understand division. It uses a grid to model the division problem, as shown in Figure 2-11. Students specify the values of the problem using arrows for the dividend and a slider for the divisor. The tool shows them the appropriate number of grid squares filled in, with the number of rows corresponding to the divisor and the number of columns corresponding to the quotient. Any squares that are part of the remainder will be displayed in red, while the rest of the squares are displayed in blue. The answer is automatically revealed on the x-axis of the grid. Students can use the system to quiz themselves on division: The tool gives them a division problem and students have to adjust the slider to make the number of rows reflect the divisor. Then they must enter the divisor and remainder into the empty textboxes in the equation, which they can determine from the picture, and the tool will tell them whether or not they are correct. This tool also does not allow the students to have much interaction with the visual model, aside from specifying the values. We believe that direct interaction with the visual model is an important part of the educational

value of these types of tools, and this is reflected in our design outlined in the following sections.

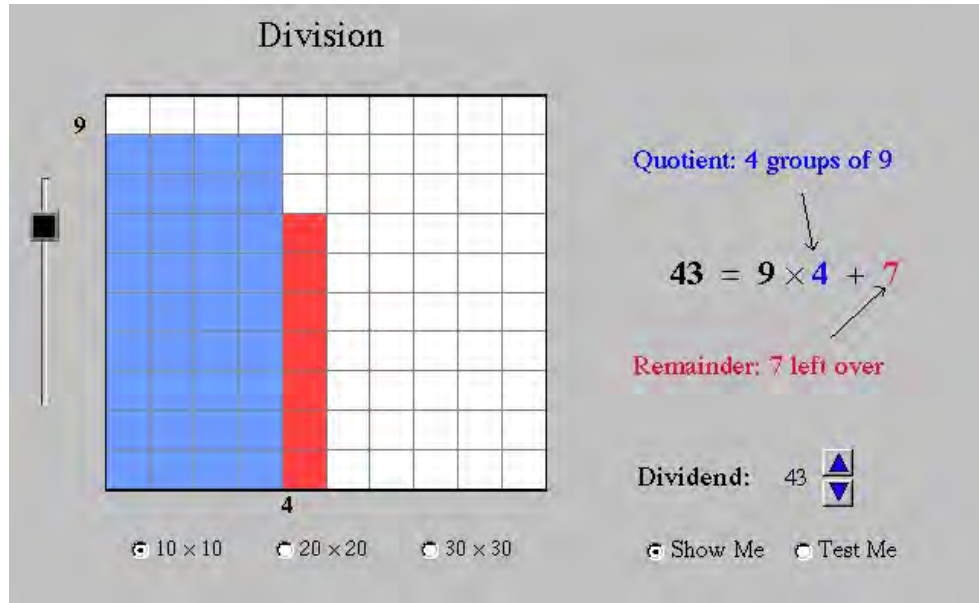


Figure 2-11: The National Library of Virtual Manipulatives division tool

# Chapter 3

## Division Tool

The solution we implemented, originally referred to as the "fuzzy factor card", is a variation on the factor card. The idea is to create factor cards that have one edge represented as "fuzzy", i.e. ill-defined, in order to portray the uncertainty of one of the dimensions of the array represented by the card. Students could use these "fuzzy factor cards" to solve division problems by snapping smaller arrays inside of them until the uncertain dimension was determined. We selected this idea because it leverages the understanding of arrays as a model for multiplication that students using the software would have already developed from using the array tool for multiplication. It thus provides a strong visual model for the relationship between multiplication and division. This model does not reflect partitive division as well as quotative, since it uses groups of known size—represented as arrays—instead of a specified number of groups, but we hope that it will promote more abstract thinking about division in general and could thus be applied to partitive problems as well. There are two versions of the division tool, one designed for use by students who are first learning division and could benefit from an explicit visual representation of a dividend and remainder, and one for students who have a firmer grasp on the concept of division.

### 3.1 Division Tool, Basic

To access the tool, students tap the "Division Tool" button on CLP's command ribbon to bring up a window that allows them to enter the product and one of the factors, shown in Figure 3-1. When the student taps on "Enter", a division template appears on the page. Figure 3-2 shows an example of a division template representing the problem 55 divided by 7. The 55 in the center represents the dividend and the 7 on the side represents the divisor. As with other objects in CLP, the division template

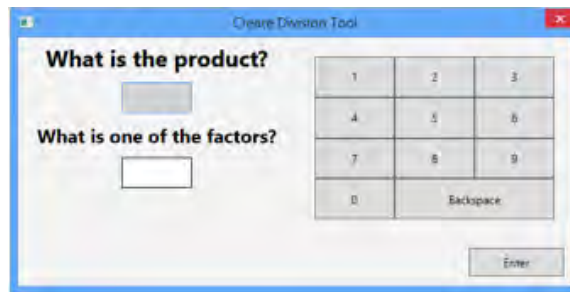


Figure 3-1: The creation window for the division tool

has a variety of adorners, which can be used to move, resize, or delete it. The adorer unique to the division template is the minus adorer in the bottom left, which removes the most recently added array. These adorners are shown in Figure 3-2.

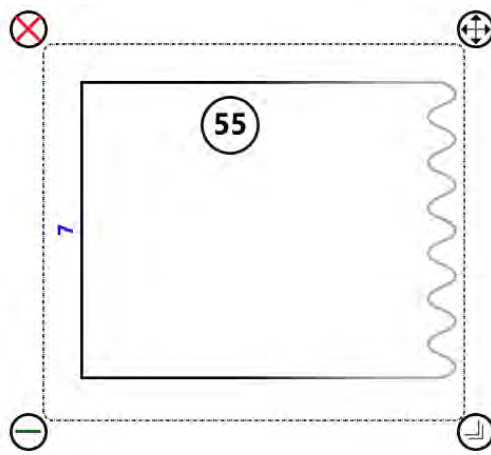
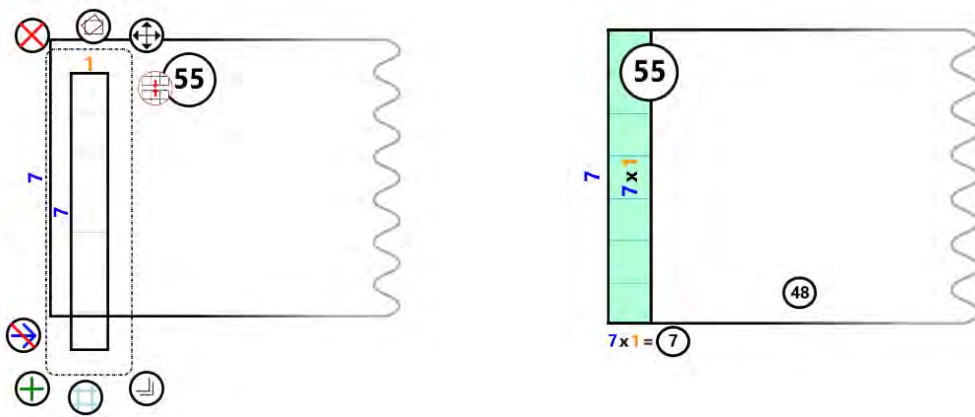


Figure 3-2: A division template representing the problem 55 divided by 7; adorners are used for operations on the template

Students solve a division problem by creating a division template that matches the problem, then creating and snapping arrays into the template. The arrays representing groups of factor pairs are snapped inside the division template using the drag and snap adorning on the right edge of an array, analogous to snapping two arrays together, as described earlier. If the array edge is close enough to the right edge of the other snapped in arrays, or the left edge of the division template if no arrays have been snapped in, the array will snap inside and become part of the division template, as shown in Figure 3-3. A variety of values then appear inside the division



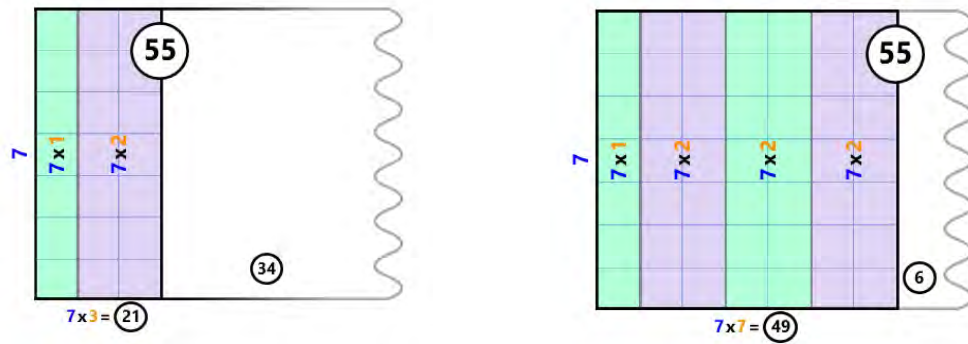
(a) Students drag the array into the template (b) The array snaps into the template

Figure 3-3: A division template with an array being snapped inside

template. The appropriate factor pair appears inside each snapped in array. The equation at the bottom represents the size of each group multiplied by the number of groups snapped in so far, and the circled number is the total area of the snapped in arrays. The circled number in the empty portion of the division template represents the current value leftover, i.e. the amount left to reach the dividend. When more arrays are snapped in, the shading colors alternate in order to delineate the adjacent arrays, as shown in Figure 3-4.

If a student tries to snap in an array with a different number of rows than the division template, or an area larger than the current amount leftover, the array will not snap in, and borders of the division template will flash red to indicate that the array

could not be snapped. If the number of columns matches the number of rows in the template, the student can rotate the array using the rotate adorning and try the snap operation again.



(a) A division template with two arrays

(b) A full division template

Figure 3-4: Division templates with arrays snapped inside

Students continue snapping arrays into the division template until the amount left over is less than the factor. They then can find their answer as the orange number in the equation on the bottom, or the number of columns of snapped in arrays. Since the tool reveals the answer for the student, it is meant as an introductory tool to help students develop an understanding of division, which they can use later to solve problems on their own without the tool.

## 3.2 Division Tool, with Remainder Representation

We designed an alternate version of the division tool, which contains a display of tiles to represent the current leftover value. This version is helpful for students who are new to the concept of division or those who are struggling with the concept of remainders. Students use this version by tapping the "Division Tool With Tiles" button on the command ribbon, and then enter the appropriate values in the division template creation window. In this version, a group of blue tiles is placed on the page, as shown in Figure 3-5. The tiles represent the current amount leftover, initially equal to the

dividend. The tiles match in size and color the cells in the arrays snapped into the division template. This matching reinforces the idea that the tiles will be "put into" the division template and that any that do not fit represent the remainder. This group of tiles can be moved independently of the division template, so students can manage their screen space, but it cannot be deleted or resized.

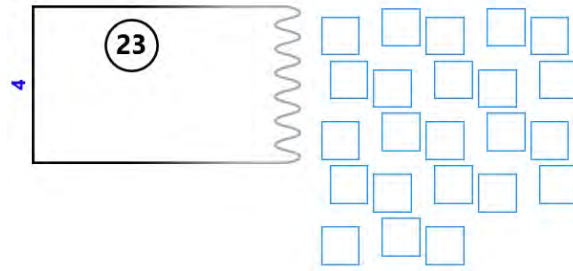


Figure 3-5: A division template with tile region displayed representing the problem 23 divided by 4

This "putting tiles" into the division template is visualized in the following way: When an array is placed on the page, a number of tiles corresponding to the number of cells, i.e. area, of the array will turn black, as shown in Figure 3-6. When the array is snapped in, the black tiles will disappear to signify that they are now in the division template, as in Figure 3-7. When the division template is full, the number of tiles leftover corresponds to the remainder of the problem, as in Figure 3-8. The number of columns of the snapped-in arrays corresponds to the quotient, as before.

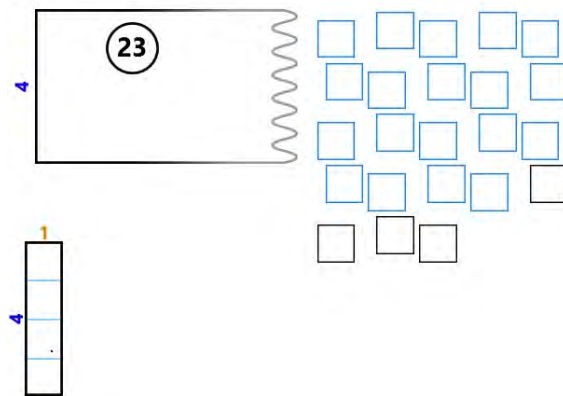


Figure 3-6: Four tiles corresponding to the area of the array have turned black

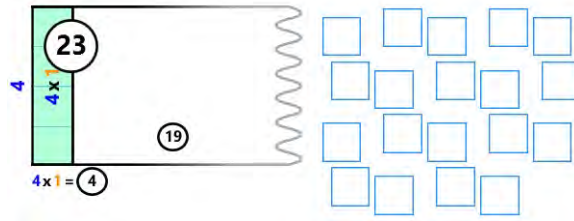


Figure 3-7: After the array snaps in, the four tiles disappear

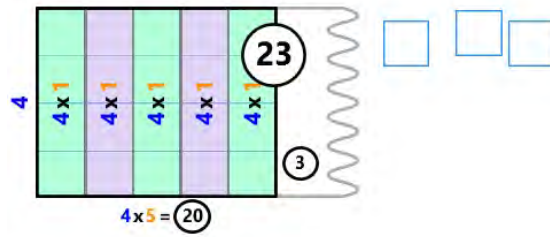


Figure 3-8: A division template with tiles at the completion of the problem  $23 \div 4$ , which has a solution of 5 with remainder 3



# Chapter 4

## Design

We went through a variety of possible design choices before arriving at the design presented in the previous section. Throughout this process we received feedback from educational researchers, teachers, and children with whom we tested the tool.

### 4.1 Design Objectives

Throughout the design process we kept several design objectives in mind. We wanted the design to be consistent with students' understanding of arrays in order to provide a strong visual model. We also wanted the computation performed by the system to assist student understanding without oversimplifying the process to a point where students did not need to do any mathematical thinking. Our goal was to help students develop a deep understanding of division.

### 4.2 Design Issues

#### 1. The Array Model

There were a variety of ideas that we considered during the design of this tool. The first, which was considered early on before committing to implementing this tool, was the decision to represent division using the array model. This decision was made because we thought leveraging students' understanding of arrays would be an effec-

tive way to help students visualize a complicated concept. Since the array model is useful for helping students understand multiplication [5] we believe it has significant potential to help students understand division as well.

## **2. Representing approximate area**

We also considered how to represent the total in a division problem as an area. We considered representing the exact area, or representing only the area of the parts the students had added so far, but we thought that displaying an approximate area would be most effective at communicating the fact that one dimension was unknown.

A major design challenge was determining how to make the area appear ill-defined. In the design phase we drafted a variety of different possibilities for the "fuzzy" edge, including the wavy line that was shown in Section 3, omitting the right edge entirely, portraying the right edge as a dotted line, or using shading to make one side of the array appear to fade into the background. We decided to implement the wavy edge, but after some initial testing this choice was revisited, as will be discussed below.

Another possibility we considered was updating the width of the card after a calculation was complete, and making the edge appear solid instead of fuzzy. If a student snaps in the appropriate area of smaller arrays, should the fuzzy edge disappear to represent this situation? Should the student be responsible for acknowledging that the solution is complete and initiate this snapping into place? Initially we decided against replacing a fuzzy edge with a straight edge, but this question was revisited later on, and we decided that the fuzzy edge should disappear if the total has been reached with no remainder, as this is a more accurate representation of the array model. A full division template with the fuzzy edge removed is shown in Figure 4-1.

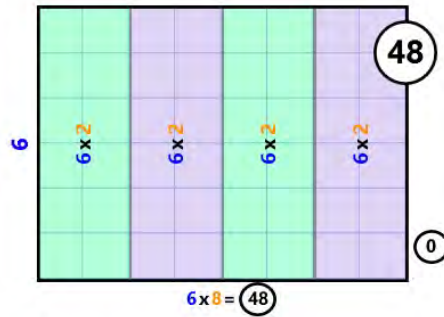


Figure 4-1: The fuzzy edge disappears if the value leftover goes to zero

### 3. Adding and Removing Arrays

We considered a variety of different methods for enabling students to find an answer by adding arrays into the division template. In an initial round of brainstorming, we decided that a two step process of creating and then snapping in arrays might be too tedious for students, so we planned to add an operator that would let students add arrays by tapping on a plus icon. We decided, however, that there was educational value in students having to be deliberate about choosing dimensions for an array to be added. With a single plus operator, the students could too easily ignore the size of the groups they were adding. So we returned to the initial idea of snapping the arrays in, demonstrated in Figure 4-2, as it forced students to be mindful about the process of adding up groups of factor pairs to a total. We considered the possibility of having arrays snap in any time they were dragged over the division template, but decided for consistency to make the snap work in the same way as when two arrays snap together, by dragging an array into the location where its edge touched the edge of the division template or most recently snapped-in array.

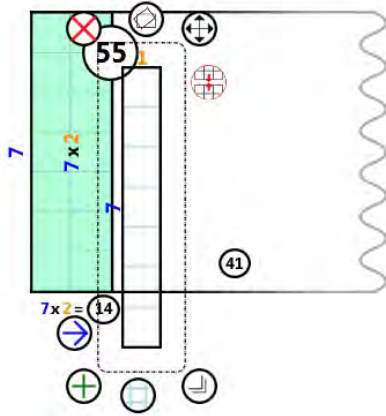


Figure 4-2: A division template with an array being snapped in

We decided that students might want to be able to remove arrays, initially so that they could add more than the total and then subtract after realizing their mistake. We came to the conclusion that going past the total and subtracting would be too complex for students, but they might still want to be able to remove arrays in case they decided to change their strategy in the middle of a problem, e.g. adding groups of five instead of groups of one. We debated the possibilities of adding an "undo" specifically for the division template or allowing students to erase the array dividers. We decided to add an undo adorer in the shape of a minus symbol, which can be seen on the bottom left in Figure 4-3. Tapping on that adorer allows students to remove their most recently added array. The action is easy to comprehend and allowed us to avoid the complicated question of what happens to the two arrays on either side of an erased divider.

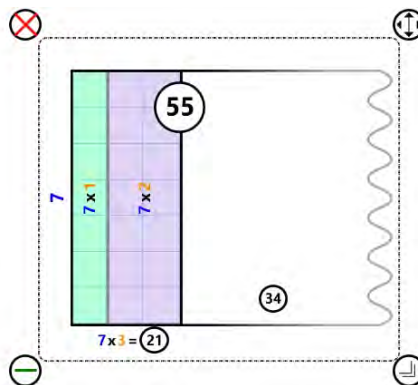


Figure 4-3: The minus adorer on the bottom left can be used to remove arrays

## 4. Computation and Totals

Another design issue we examined was how much calculation should be performed by the software. When a student is inserting arrays into the division template, the software could automatically increment the array area so far, while decrementing the remainder. We could also imagine the student having to perform this calculation on their own or having the system provide some sort of reminder. Another important question to examine was whether or not displaying both of these totals—the area snapped in and the area remaining—would be confusing to the student, and if so which of the two would be more useful to display.

We decided that it would be best to display both totals, and also display the dividend inside the array in larger font, in order to help students keep the original problem in mind. We thought displaying the total area of arrays snapped in as well as the area remaining was important because it reflected the idea of counting up and counting down at the same time, and it helped connect these two interpretations. We added color variation to help distinguish between the different totals, as shown in the figures below. There was concern that displaying all the totals could be confusing, but after several redesigns of the layout, shown in the following section, we came up with a design that we felt better conveyed the meaning of each total. The first redesign added factor pairs inside each snapped in array and an equation at the bottom to reinforce the meaning of those areas. In a later iteration we moved the dividend to the edge of the snapped in arrays, since we thought this better emphasized the fact that it represented the area on both sides of the dividing line.

### 4.3 Designs

#### 1. Preliminary design

We initially designed the division tool as a factor card with a wavy right edge. A

mockup of this design can be seen in Figure 4-4. Smaller arrays could snap inside, as described before. We planned to display the total area of arrays snapped in and the remaining area to reach the dividend inside the appropriate portions of the division template, as well as the divisor on the left edge and the number of groups snapped in on the top edge. We also planned to display a question mark on the top edge of the empty portion of the division template, to signify that the number of groups left to add was unknown. In this design the wavy edge would remain even when the total was reached.

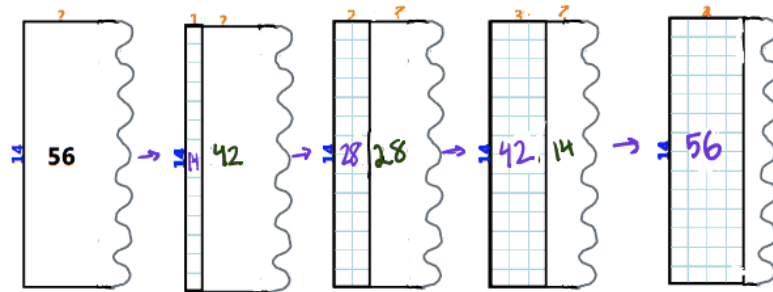


Figure 4-4: A preliminary design for the division template. In this example, smaller arrays are snapped inside the division template until the total is reached

## 2. First Prototype

Before implementing the first prototype, shown in Figure 4-5 we decided it was important that the dividend always be displayed, as it is important for students to keep in mind the problem being solved. We displayed the dividend in the center of the division template with a yellow background. We decided to display the other totals in a smaller font near the bottom of the array. We added labels for the width of each snapped in array at the top, as well as a label that displayed the total width of all snapped in arrays at the bottom. We also implemented versions that kept this total, which would eventually become the answer, hidden, in order to force students to do some of the calculation. Another version we implemented reversed the placement of the total number of array columns and the individual array widths, as we wanted to evaluate which aligned best with our mathematical model and was most consistent with our array tool. We used a variety of different colors to distinguish between each

type of label. We also shaded the snapped-in arrays in grey in order to distinguish them from the rest of the division template.

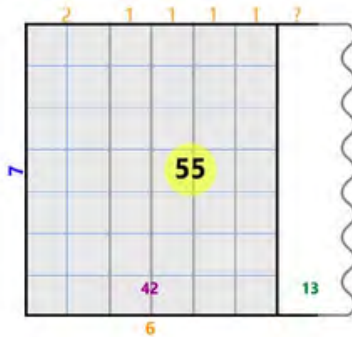


Figure 4-5: The first prototype of the division tool

### 3. Redesign

After building the initial prototype and receiving input from educational researchers, we came up with a new design which we thought better represented the meaning of all the totals and provided students with more intuition about the model. The mockup we designed can be seen in Figure 4-6. We added alternating colors in order to distinguish between the snapped in arrays. In the initial implementation of this version, the dividend was always centered in the division template and the fuzzy edge always remained.

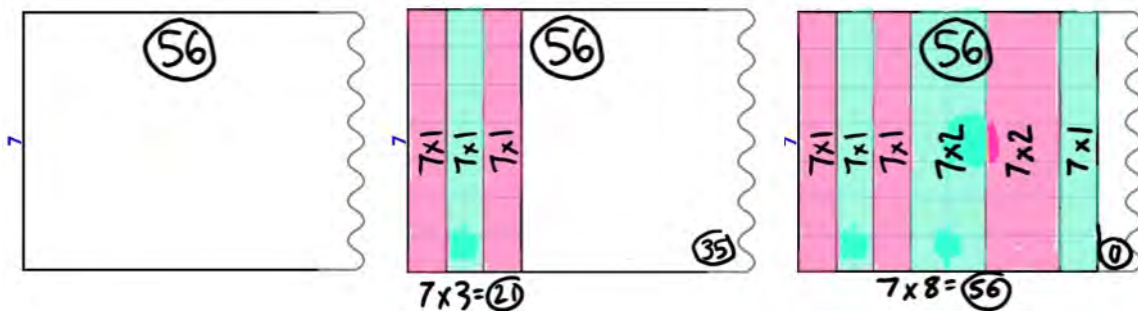


Figure 4-6: A draft of first redesign of the division template

### 4. Final Design

The final design, shown in Figures 4-7 and 4-8, is very similar to the redesign shown above, with a few changes we made after testing with students. We decided to display

the dividend on the edge of the snapped in arrays, or in the middle when the division template is empty, to clarify that it was meant to represent the area of the entire object. We also decided that the fuzzy edge should disappear when the template is full and there is no remainder, and stay when there is a remainder.

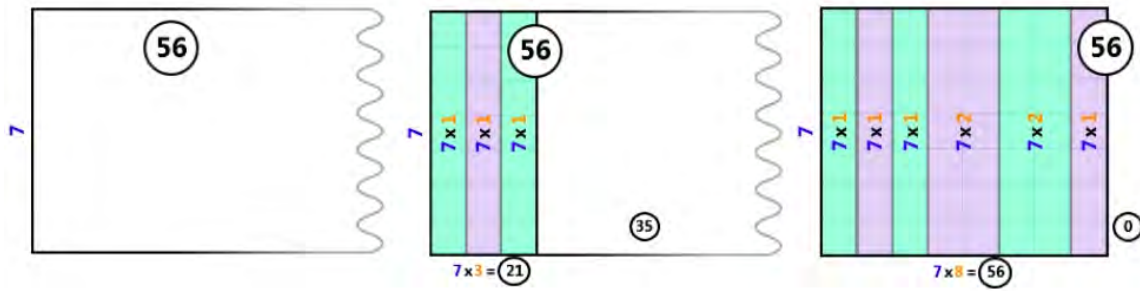


Figure 4-7: The final version of the division template

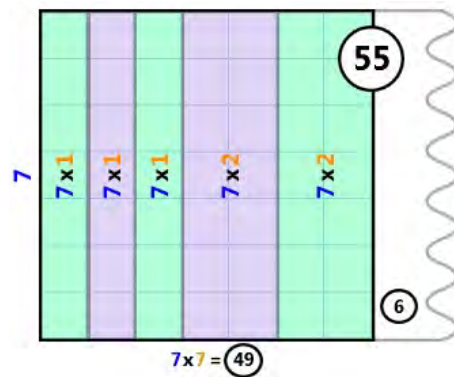


Figure 4-8: The final version of the division template with a remainder



# Chapter 5

## Testing

Throughout February, March, and April of 2014, we tested various prototypes of the division tool with several sets of students. We gained a lot of insight about how the tool performed with students and used that knowledge to improve upon the design.

### 5.1 Fuzzy Edge

In the initial round of testing, we worked with relatively advanced students who seemed to catch on to the idea of the tool very quickly even though they had not worked with arrays in a while. When asked what they thought about the tool, though, they mentioned that they found the wavy edge confusing. One mentioned that the fact that the edge contained six bumps could have been confused as information relevant to the problem. Because of this feedback, we revisited the design for the right edge. In our next round of testing we tried out the original design, as well as a lighter colored but completely straight right edge, and one with no right edge and the other edges fading out to white. These alternative designs are shown in Figure 5-1. The feedback from this round, however, indicated that the new alternatives were even more confusing to students. In the straight edge design if the final result had a remaining chunk of area, which was necessary if there was a remainder, students were unsure if they had finished the problem. The open right edge was even more confusing because students could no longer visualize a definitive object. We reasoned that the wavy edge was the best option we had, and that even though some students

found it distracting initially, once they got used to the tool the wavy edge was most efficient at supporting their understanding.

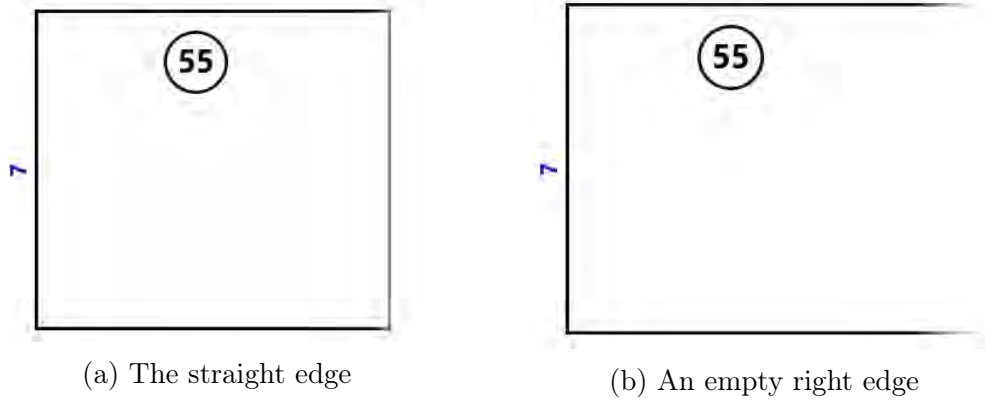


Figure 5-1: Alternative versions of the fuzzy edge

## 5.2 Tiles

Another insight we realized during testing was that students often struggled with the concept of remainder. One student we worked with, Ashley, particularly struggled with this concept. She seemed to have trouble understanding the meaning of the leftover value. While working on the problem  $48 \div 6$ , after snapping in one 1 by 6 array, she was asked what the value of 42 on the right would become after snapping another 1 by 6 array in. She was unsure about what it meant, and guessed that it would become 49. After snapping the array in, she was able to grasp that the value was decreasing, noting "they're taking away." When we got to problems with remainders, we decided to try a smaller example of  $20 \div 3$ . Before working with the division tool, she reasoned about the problem and asked if the remainder would be 1. Her reasoning was that she could count up to 21 using 3's, and so she seemed to think the remainder was the amount her total went over 20. After using the tool and snapping enough arrays to fill it up, when asked how many were leftover she figured out that the answer was 2. With a larger problem, however, she became more confused about

the remainder. When given the problem of  $40 \div 7$ , she snapped in a 7 by 5 array. She recognized that when the total was 35 she needed 5 more to get to 40, but when asked if there were 5 leftover or 2, she answered 2, reasoning, "If we're gonna stop at 40 we're gonna have 2 more left because it's 7."

This interview helped us realize that for some students using our tool, displaying the value of the remainder in the open portion of the division template was not a strong enough representation. We thought it might help to have a more concrete way of showing what is actually leftover and how that relates to the rest of the problem. We brainstormed a new idea that involved displaying small squares, referred to as "tiles," for each number leftover, and having the tiles disappear as the student snapped in arrays. The squares were meant to be a more explicit representation of the cells in an array. After testing we learned that this configuration was a very effective way for students to visualize the concept of remainders.

### 5.3 Student Use

Overall we found that the tool was very helpful for supporting students' understanding of division, when they already had a sufficient understanding of arrays as a model for multiplication. One student we worked with, Darcy, had not worked with arrays as a model for multiplication and had a lot of trouble with the tool. When we got to the first division problem,  $48 \div 6$ , she knew the answer already and put down an appropriate division template, but struggled to create and snap in appropriate arrays. She first attempted to put down an 18 by 6, which was in the wrong orientation to snap in and also would have been too large to fit. At various points in the problem she had trouble interpreting the numbers represented in the tool. When 12 was leftover, for example, upon being asked how many groups of 6 she could make, her answer was 4. We asked her what was confusing about the tool and she answered that she "never did it that way." We had anticipated that students with whom we

would be working would have more familiarity with arrays, but we often had to introduce them to the concept, which hindered our ability to determine if this tool could successfully build upon a more developed understanding of arrays as we had intended.

For another student who seemed less experienced with arrays, Zelda, the tool still appeared to help her understanding of division. She was able to work with the tool relatively easily despite having little experience with division. Throughout her work with the tool, she began to use progressively larger array sizes. After completing the problem  $96 \div 8$  using arrays of width 2, 3 and 4, shown in Figure 5-2, she noted that she could have instead used a 10, as 12, the answer to the problem, could be broken into 10 and 2. This observation indicates that the tool helped her to gain some insight about more efficient division strategies, in particular using groups that align well with the decimal system.

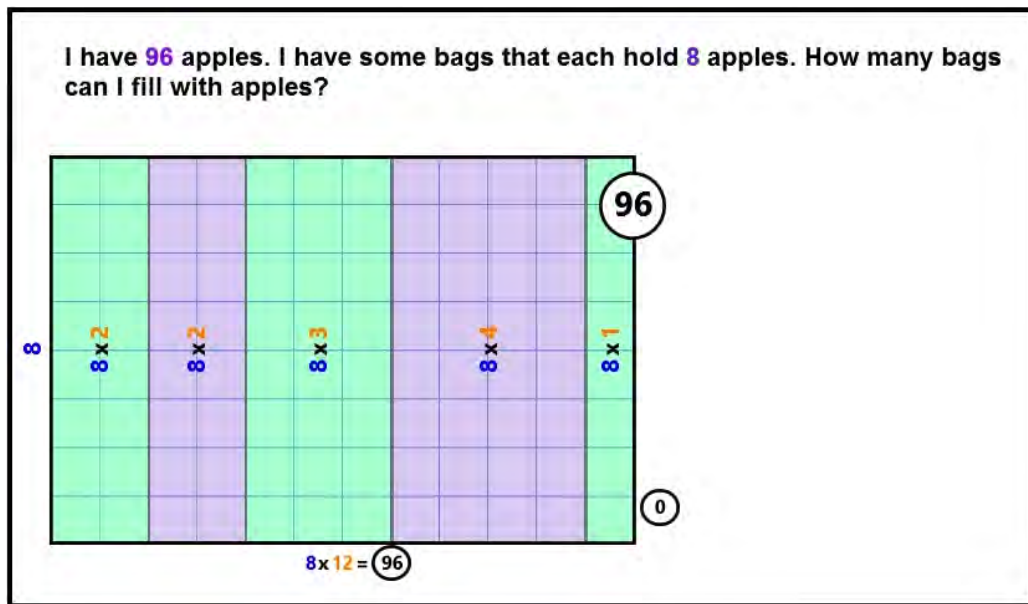


Figure 5-2: Zelda's solution to the problem  $96 \div 8$  using the division tool

For students who had used arrays as a model for multiplication, the division tool was generally understandable and helpful. We worked with one student, Adrian, who was more advanced and was able to figure out most of the multiplication facts before using

the division tool to model them. When we gave him a problem with a remainder,  $98 \div 8$ , however, he said, "This one I need the tool," and was then able to use the tool to figure out what the remainder would be. Adrian's solution to this problem is shown in Figure 5-3.

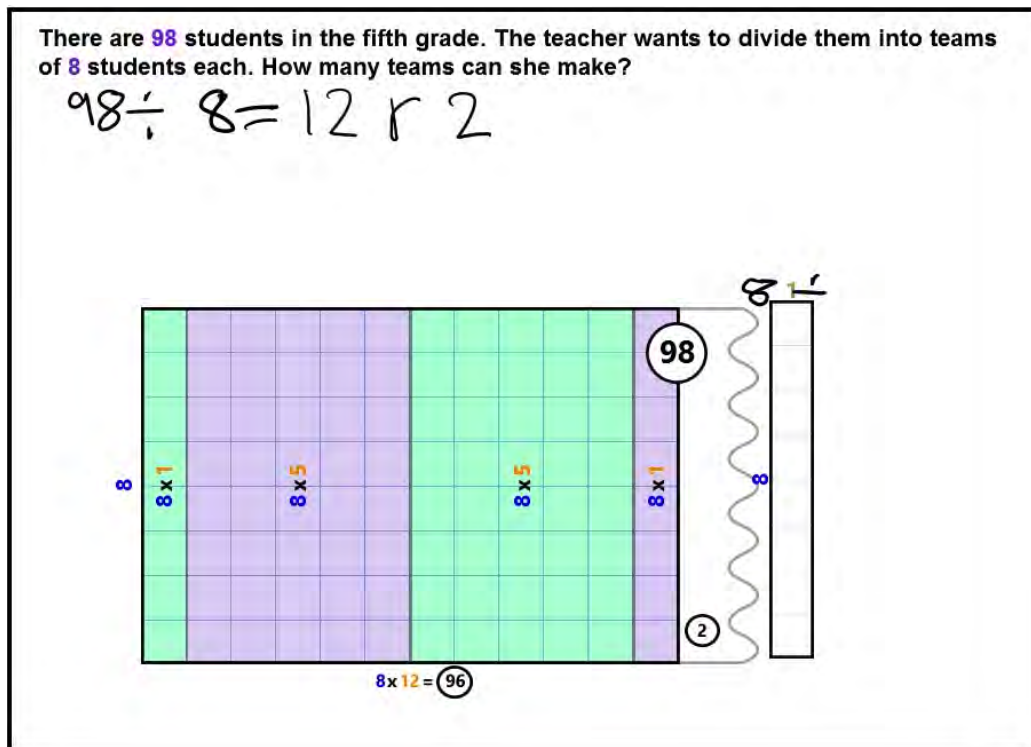


Figure 5-3: Adrian's solution to the problem  $98 \div 8$  using the division tool

After each interview we asked if the students found the tool helpful or if it would be helpful for other students starting to learn division, and most answered that it was. Students cited various reasons for the division tool being useful, including that it helped them see the problem and keep track of the different numbers. One student described the tool as having a "fun interface."

We also had the opportunity to test the tool in a classroom setting. The students had been working with the array tool for the past few weeks, so many of them had a strong intuition for how to use the division tool. While the tool was being introduced to the class using an example problem, one student quickly predicted that the orange

number in the bottom equation would be the solution to the division problem. This observation showed that the tool was fairly intuitive for him to understand. The tool seemed particularly helpful for two students, Natasha and Katie, who usually work as a pair with Natasha assisting Katie. When the tool was introduced, Natasha had a strong intuition for what all of the numbers meant and helped explain the tool to Katie. Initially Katie misunderstood what the solution would be and thought it would be the total area of snapped in arrays, but Natasha told her to look at the orange number representing the number of groups. Once they started their individual work, solving the problem  $45 \div 9$ , Katie immediately realized that she could fill the division template with a single 5 by 9 array. This realization indicated that she had a strong understanding of this model. In general it seemed that students for whom the array tool was useful also could easily work with the division tool and found it useful as well.

## 5.4 Teacher Feedback

We received feedback from teachers that this tool has a lot of potential to help students understand division. One teacher remarked that this tool very closely models how the teachers hope to get students thinking about division, by using combinations of factor pairs that they already know to get them close to the total. She thought that this tool would be especially helpful for students just beginning to work on division, since we eventually want them to be able to keep track of the values on their own without the tool, which is in line with our original intentions. She also noted that the tile tool was helpful because it visualized the concept of repeated subtraction. The color coding stood out as something she thought was particularly useful, because it would help reinforce the relationships between the numbers.

One place for improvement she noted was that the fuzzy edge could appear a bit confusing, because it was on the vertical length, i.e. the number of rows, that was

known. Our intention was to make the horizontal length of the array, i.e. the number of columns, unknown, but she noted that the wavy edge itself might be misinterpreted as representing the unknown value, giving off the false impression that the number of rows was the unknown instead of the number of columns.





# Chapter 6

## Automatic Analysis of Student Work

CLP can automatically analyze certain types of student work in order to provide feedback to teachers. This is accomplished by creating what we call tags when a student performs a specified action or when student work is submitted and then analyzed by the software. The teacher can view these tags while looking at students' work, allowing him to quickly gain insight about his students' understanding or search for student work that fits a particular pattern.

After implementing the division tool, we added several tag types so that the software could analyze work done with the division tool. These tags include:

- The *correctness tag*, which reports whether the division template is entirely filled with arrays, partially filled, or empty.
- The *representation correctness tag*, which reports if the values entered are consistent with those described in the problem.
- The *strategy tag*, which gives information about the student's strategy, such as whether she filled the division template with only one array, all arrays of width one, or used groups of landmark numbers such as 5 or 10.
- The *failed snap tag*, which reports whether the student attempted to snap an array that didn't work, either because the dimension or orientation was incorrect, or because the division template did not have enough room left for the new array. The tag specifies if the incorrect dimension is equal to the product,

as we found this to be a common mistake. It also records whether a failed snap has happened only once or multiple times.

- The *added incorrect array tag*, which reports whether an array was placed on the same page as a division template and it could not have been snapped in for any of the reasons above, e.g. if a 7 by 1 array were placed on the same page as a division template with 6 rows.
- The *deleted division template tag*, which reports whether the student created and then deleted a division template and, if so, what its dimensions were.

The strategy tag type has a variety of possible values. The first strategies we included were using all groups of 1, using only one array, and using groups of 10. We later decided it would be useful to know if a student had used groups of 5, and if she had repeated any group size other than those previously mentioned. In the original implementation, the tag for groups of 10 and 5 checked only the first array snapped in, since it was assumed that if a student was using a tens strategy she would start with the ten and add smaller groups later. We later decided it would be useful to know if a student had used a group of 10 or 5 anywhere in her work.

The tags that are generated as the student is working, such as the failed snap tag, are a new type of tag. While working with students we realized that their misunderstandings about how to use the tool were very important, but were not often visible in a final answer. We decided it would be a good idea to add a tag to report when the student attempts to use the tool incorrectly, such as trying to snap a 1 by 10 array into a division template with 7 rows. We later realized there was an important distinction between snapping in an incorrect array and placing one down but realizing it would not snap, so we decided to add a tag for this situation as well. While working with students, we saw some students try to snap arrays that would have snapped if rotated first, such as a 1 by 7 array into a division template with 7 rows, and realized this situation indicated a different type of misunderstanding than snapping an entirely

wrong dimension. In order to keep this distinction, we added a value type for this tag to reflect the possibility of attempting to snap an array with the wrong orientation.

We saw several students create division templates that were inconsistent with the problem description. If the student deletes that division template the teacher will not be aware of this mistake when looking at the student's final answer. We thought knowing this could reveal insight about the student's misunderstanding, so we added a tag to keep track of any division templates which were deleted, and the dimensions that were specified.



# Chapter 7

## Future Work and Contributions

### 7.1 Further Testing

There is still a lot more that can be learned about how this division tool works for students. So far we have only had a chance to see students work with the tool over a period of several hours, so it would be useful to observe how the tool supports a student's understanding of division over a longer period of time. In particular, it would be useful to learn whether practice with the tool allows students to solve division problems more efficiently without the tool. It also would be useful to study whether the tool encourages students to be more efficient with division strategies by breaking the problem up into large chunks or if it simplifies the process too much by performing all the calculation, thus encouraging students to choose smaller groups. It was difficult to learn anything about this as we only saw students using the tool for a short period of time, so many of them worked with smaller groups because the tool was still rather unfamiliar. We also would be interested in studying whether displaying all the totals supports student understanding better than displaying a subset of the totals.

## 7.2 Improvements

Another portion of work that can be done is improving upon the current tool. We could try more variations on how the numbers are displayed and test which variation seems to improve student understanding the most. We also could add versions that have the student do some or all of the calculations on their own, such as writing out the equation by themselves. We could work on alternate visual representations, especially for the fuzzy edge, since several students found the edge confusing.

## 7.3 Contributions

This thesis developed a tool to help elementary school students better understand division. The tool uses students' prior understanding of arrays to represent the concept of division in a familiar way. It helps students strengthen their understanding by allowing them to interact directly with a visual model. The results of our work with students indicate that this tool has the potential to make a significant impact on students' proficiency in division.

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